RIGIDITY AND ABSENCE OF LINE FIELDS FOR MEROMORPHIC AND AHLFORS ISLANDS MAPS

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Abstract. In this note, we give an elementary proof of the absence of invariant line fields on the conical Julia set of an analytic function of one variable. This proof applies not only to rational as well as transcendental meromorphic functions (where it was previously known), but even to the extremely general setting of Ahlfors islands maps as defined by Adam Epstein.

In fact, we prove a more general result on the absence of invariant *differentials*, measurable with respect to a conformal measure that is supported on the (unbranched) conical Julia set. This includes the study of cohomological equations for $\log|f'|$, which are relevant to a number of well-known rigidity questions.

1. Introduction

An *invariant line field* of a holomorphic function f is a measurable field of non-oriented tangent lines that is supported on a completely invariant set of positive Lebesgue measure. The *no invariant line field conjecture* states that, with the exception of flexible Lattès mappings (see below), rational functions do not support invariant line fields on their Julia sets. Mané, Sad and Sullivan [MSS] showed that this conjecture is equivalent to the density of hyperbolic maps in the space of all rational functions (a conjecture that can be traced back to Fatou; compare [McM, p. 59]). Correspondingly, the problem—which remains open even for quadratic polynomials, $f(z) = z^2 + c$—has received significant attention.

However, it is well-known that such line fields do not exist when there is sufficient expansion. In particular, the absence of line fields on the *conical* (or *radial*) set—on which there is nonuniform hyperbolicity—was proved by McMullen in [McM]. This result was generalized to arbitrary transcendental meromorphic functions by Rempe and van Strien [RvS], extending earlier partial results by Graczyk, Kotus and Swiątek [GKŚ].

These original proofs use orbifold theory and are somewhat involved. In [MM], a rather elementary proof of McMullen's result was given (and the argument applies equally to transcendental entire functions). In this note, we extend this reasoning to the vast class of *Ahlfors islands maps* introduced by Epstein [Ep2]. These are nonconstant holomorphic functions $f : W \to X$, where X is a compact Riemann surface and W is a nonempty open subset of a compact Riemann surface Y, that satisfy a certain function-theoretic transcendentality condition near every point of ∂W. (See Section 2 for the precise definition.) For the purposes of iteration, one is interested in the case where $Y = X$; the set of all Ahlfors islands maps with this property will be denoted by $\mathcal{A}(X)$. Such a function is *non-elementary* if it is not a conformal automorphism. We note that this class contains all rational functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$,

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all transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$, the functions of [BDH] and Epstein's *finite-type maps*. The later class includes all (iterated) parabolic renormalizations of rational maps.

As mentioned above, our results concern the *conical* (or *radial*) Julia set. There are several inequivalent definitions of this set in the literature. For our purposes, we denote by $\Lambda_c(f)$ the *(branched) conical set* of f, that is, the set of points in the Julia set for which it is possible to pass, using iterates of f, from arbitrarily small scales to a definite scale by a map of bounded degree d. The set of points for which we can take $d = 1$, i.e. the set of points of $\Lambda_c(f)$ that have univalent blow ups, is denoted $\mathcal{J}_c(f)$ and called the *unbranched conical Julia set*. See Section 3 for the formal definitions.

Theorem 1.1. *A non-elementary Ahlfors islands map $f \in \mathcal{A}(X)$ does not support an invariant line field on its conical set $\Lambda_c(f)$ unless f is either a linear torus endomorphism or a flexible Lattès map on the Riemann sphere.*

In particular, our theorem applies to transcendental meromorphic functions, and thus provides an elementary proof of the result from [RvS]. We also emphasize that the argument from [RvS] does not extend to our more general setting of Ahlfors islands maps. As usual, the proof begins by a renormalization argument that shows that such a line field would have to be *univalent* near some point of the Julia set, and the main step of the proof consists of providing a classification of maps that possess such line fields (Theorem 6.1).

Similarly as in [My], our way of establishing Theorem 1.1 allows us to handle a much more general framework. For the purpose of this introduction, let us consider the case of the unbranched conical set $\mathcal{J}_c(f)$, which allows for simpler statements.

First of all, Theorem 1.1 only applies to maps whose Julia set has positive Lebesgue measure. It is natural to replace Lebesgue measure by other dynamically significant measures, such as invariant Gibbs states and conformal measures. In the setting of rational functions, Fisher and Urbański [FU] studied invariant line fields that are not defined on a set of positive Lebesgue measure but on the support of some conformal measure. (More precisely, they work with invariant Gibbs states, but these are equivalent to corresponding conformal measures). They obtained a statement analogous to Theorem 1.1, the only difference being that there is a longer list of exceptional functions. This work has then been used to establish ergodicity of the scenery flow of a hyperbolic rational function ([BFU]).

An equivalent way of stating that a function f, defined on a subset of the complex plane, has an invariant line field is that $\arg f'$ is cohomologous to a constant. More precisely, there is a measurable function $\mathcal{L}$ such that

$$\mathcal{L}_{f(z)} = \mathcal{L}_z + \arg f'(z) \mod \pi \quad \text{a.e.} \tag{1.1}$$

There is another circle of rigidity questions that relies on a different cohomological equation. This time, instead of $\arg f'$, the crucial point is to study when $\log |f'|$ is cohomologous to a constant. This is a central point of the measure rigidity problem, initiated by Sullivan [Su1] and developed further by Przytycki-Urbański [PU]; see also the paper [KU] by Kotus-Urbański where this problem has been investigated for a class of transcendental meromorphic functions.

Another striking example is Zdunik's paper [Zd] which, in combination with a result by Makarov [Mk] on harmonic measure, shows that Julia sets of polynomials are fractals. More precisely, if f is a polynomial with connected Julia set $\mathcal{J}(f)$ then either $\mathcal{J}(f)$ is a circle or line segment (in which case $f(z) = z^{\pm d}$ or f is a Chebyshev polynomial) or the Hausdorff

dimension $\mathrm{Hdim}(\mathcal{J}(f)) > 1$. The heart of this dichotomy results from stochastic limit theorems (central limit theorem and the law of iterated logarithm) and the smooth exceptional examples correspond exactly to the case where the variance equals zero. This last condition is equivalent to the statement that a cohomological equation for $\log|f'|$ holds μ_f-a.e., where μ_f is the maximal entropy measure (which, for a polynomial, coincides with harmonic measure of the basin of infinity).

We give a unified approach to all of these problems. In order to do so, and in order to make sense of the questions on an arbitrary Riemann surface X, we will recast the problem in terms of *invariant differentials*. Recall that an (m,n)-differential, where $m,n \in \mathbb{Z}$ are not both equal to zero, takes the form $\mu(z)dz^m d\bar{z}^n$ in local coordinates. For example, an invariant line field can be identified with an invariant Beltrami (i.e., (-1,1)-) differential $\mu\, d\overline{z}/dz$ with $|\mu| = 1$ (see [McM, p.47]). Also, the statement that $\log|f'|$ is cohomologous to a constant corresponds to saying that some $(1,1)$-differential is invariant *up to a multiplicative constant.*

For a more detailed discussion of these concepts, and of conformal measures, we refer the reader to Sections 4 and 5. Note that the surface X comes equipped with a conformal metric of constant curvature: the spherical metric on the sphere, a flat metric on a torus, or the hyperbolic metric otherwise, and it is this metric that will be used in the notion of a conformal measure.

In this setting, we generalize Theorem 1.1 to obtain the following result.

Theorem 1.2. *Let $f : W_f \to X$ be a non-elementary Ahlfors Island map, with $W_f \subset X$. Suppose that, for some $\kappa > 0$ and $t \geq 0$, there is a $\kappa\|f'\|^t$–conformal probability measure ν with $\nu(\mathcal{J}_c(f)) = 1$.*

If there exist $(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}$ and a ν-measurable (m,n)-differential that is f-invariant up to a multiplicative constant, then one of the following holds.

(a) *$X = W_f$ is either the Riemann sphere or a torus, and f is conformally conjugate to a (not necessarily flexible) Lattès mapping, a linear toral endomorphism, a power map, or a Chebyshev polynomial or its negative.*
(b) *$m+n = 0$ and there is a finite set $A \subset \mathcal{J}(f)$ such that $\mathcal{J}(f)$ is locally a 1-dimensional analytic curve near every point of $\mathcal{J}(f) \setminus A$. In particular, ∂W_f is a totally disconnected set.*

Remark 1.3. The condition $\nu(\mathcal{J}_c(f)) = 1$ is automatically fulfilled for every ergodic invariant measure ν of a rational function having strictly positive Lyapunov exponent (see [L]). In particular, this is the case for the maximal entropy measure and the equilibrium states mentioned above.

For a large class of meromorphic functions, measures supported on $\mathcal{J}_c(f)$ were constructed in [MyUr] that are conformal with respect to a suitable conformal metric which is not the spherical metric. Theorem 1.2 also holds for these measures (see Definition 4.2 and Lemma 4.3).

Remark 1.4. An example for case (b) which is not a power map or a Chebyshev polynomial is given by $f(z) = z^2 - c$ with $c < -2$. The Julia set of this polynomial is a Cantor set contained in the real line and any constant line field defined on $\mathcal{J}(f)$ is f–invariant.

In many situations there is more rigidity in case (b), meaning that one can give the precise list of functions that fit into this case (see for example Theorem 1.2 and Claim 3.3 in [My]). Moreover:

- If f is rational, then in case (b) the Julia set must be contained in a circle on the Riemann sphere [BwEr, EvS].

- For entire functions case (b) never happens since the Julia set cannot contain isolated Jordan arcs by a result of Töpfer [Be, Theorem 20].
- If $\mathcal{J}(f)$ *is* an analytic curve and $f : \mathbb{C} \to \hat{\mathbb{C}}$ is meromorphic with only finitely many critical and asymptotic values, then $\mathcal{J}(f)$ is a straight line [Ur]. As far as we know, all known examples for case (b) are maps with $X = \hat{\mathbb{C}}$ for which $\mathcal{J}(f)$ is contained in a circle on the Riemann sphere. Examples that are not rational maps are given e.g. by members of the tangent family.

We also prove Theorem 1.2 in the case where f has an invariant differential that is *continuous* on some relatively open subset of $\mathcal{J}(f)$. (In this case there is no need to assume the existence of the measure ν.) In the setting of transcendental functions we therefore have the following statements, which appear to be new even in the case of invariant line fields.

Corollary 1.5. *For an entire function f, there is no invariant (m, n)-differential that is continuous on a relatively open subset of $\mathcal{J}(f)$. (In particular, $\mathcal{J}(f)$ does not support continuous invariant line fields.)*

Similarly, if a transcendental meromorphic function f has an invariant (m, n)-differential that is continuous on a relatively open subset of $\mathcal{J}(f)$, then $m + n = 0$ and $\mathcal{J}(f)$ is contained in an analytic curve.

2. Ahlfors islands maps

Definition 2.1. Let X and Y be compact Riemann surfaces and let $W \subset Y$ be open and nonempty. A nonconstant holomorphic function $f : W \to X$ is called an *Ahlfors islands map* if there is a finite number k such that the following property holds:

Let $V_1, ...V_k \subset X$ be Jordan domains with pairwise disjoint closures and let $U \subset X$ be open and connected with $U \cap \partial W \neq \emptyset$. Then, for every component U_0 of $U \cap W$, there is $i \in \{1, ..., k\}$ such that f has a simple island over V_i in U_0, i.e. there is a domain $G \subset U_0$ such that $f : G \to V_i$ is a conformal isomorphism.

An Ahlfors islands map is called *elementary* if it is a conformal isomorphism.

The Ahlfors islands property is implicit in Epstein's thesis [Ep1] and is stated explicitly in [Ep2]. A more detailed study is undertaken in [EO]. This class includes all meromorphic functions (by the Ahlfors five islands theorem, which gives the class its name) as well as the functions considered in [BDH] and the *finite-type maps* in the sense of Epstein. There are also many examples of Ahlfors islands maps that do not belong to these categories; compare [RR].

Given a Riemann surface X, we denote the set of all non-elementary Ahlfors islands maps $f : W \to X$ with $W \subset X$ by $\mathcal{A}(X)$, and set

$$\mathcal{A} := \bigcup_X \mathcal{A}(X).$$

If f is an Ahlfors islands map, we denote the domain of f by W_f.

The class of Ahlfors islands maps is closed under composition. (This is easy to see when the functions are elementary or their domains have nontrivial boundaries. The only non-elementary Ahlfors islands maps without boundary are rational maps and endomorphisms of the torus, and these can be treated classically.) In particular, if $f \in \mathcal{A}(X)$, then the iterates f^k belong to $\mathcal{A}(X)$ for all $k \geq 1$.

The Fatou set $\mathcal{F}(f)$ of a map $f \in \mathcal{A}$ is defined as the set of points $z \in X$ that have a neighborhood U with the property that either all iterates are defined on U and form a

normal family there or $f^n(U) \subset X \setminus \mathrm{cl}(W_f)$ for some $n \geq 0$. The Julia set $\mathcal{J}(f)$ is the complement of $\mathcal{F}(f)$ in X. All results from the basic iteration theory of transcendental entire and meromorphic functions also hold for Ahlfors islands maps. In particular, we have the following important result (proved by Epstein [Ep2] along the lines of Baker's original proof [Ba] for transcendental entire functions).

Theorem 2.2 (Baker, Epstein)**.** *Repelling periodic points are dense in the Julia set of a non-elementary Ahlfors islands map $f \in \mathcal{A}$ and the Julia set $\mathcal{J}(f)$ is a perfect subset of X.*

The key fact in the proof of the density of repellers is the following lemma (see [Re, Lemma 2.5]).

Lemma 2.3. *Let $f \in \mathcal{A}(X)$. Then there is a number k with the following property: if $V_1, ..., V_k \subset X$ are Jordan domains with disjoint closures, then there is some V_j such that every open set U with $U \cap \mathcal{J}(f) \neq \emptyset$ contains an island of f^n over V_j, for some n.*

Suppose that X is a compact Riemann surface, $W \subset X$ is open and $f : W \to X$ is analytic. If $z_0 \in W$ is a repelling fixed point of f, then there is a map Ψ, defined and univalent in a neighborhood of 0 and satisfying $\Psi(0) = z_0$, such that

$$f(\Psi(z)) = \Psi(\lambda z), \tag{2.1}$$

where $\lambda = f'(z_0)$. (The map Ψ is unique up to precomposition by a linear map.)

Using (2.1), the map Ψ can be extended to some maximal domain $W_\Psi \subset \mathbb{C}$. We refer to this maximal extension $\Psi : W_\Psi \to \mathbb{C}$ as a *Poincaré function* associated to z_0 and f. The following was first observed by Epstein.

Lemma 2.4. *If f is an Ahlfors islands map, then Ψ is also an Ahlfors islands map.*

Proof. Let V be a neighborhood of 0 on which Ψ is univalent. By definition, a point z belongs to ∂W_Ψ if and only if there is n such that $z/\lambda^n \in V$ and $\Psi(z/\lambda^n) \in \partial W_{f^n}$. Since the Ahlfors islands property is preserved under iteration, and since Ψ is univalent on V, the claim follows. ☐

One can also define *Picard points maps* or *Casorati-Weierstraß maps* in analogy to the definition of Ahlfors islands maps, generalizing these classical theorems of complex analysis instead of the Ahlfors islands theorem; compare [EO]. However, these do not seem to be sufficiently strong to obtain an interesting dynamical theory. For our purposes, the following definition will nonetheless be useful.

Definition 2.5. Let X and Y be compact Riemann surfaces, let $W \subset Y$ be open and nonempty and $f : W \to X$ be holomorphic and non-constant. We say that f has the *weak Casorati-Weierstraß property* if, for every open set $U \subset Y$ with $U \cap \partial W \neq \emptyset$, the image $f(U \cap W)$ is dense in X.

Clearly, if f is an Ahlfors islands map and U is a component of W_f, then the restriction $f_{|U}$ is also an Ahlfors islands map, and in particular a weak Casorati-Weierstraß map. We note furthermore that Lemma 2.4 holds also for the class of weak Casorati-Weierstraß mappings. The reason is, again, that the weak Casorati-Weierstraß property is preserved under iteration.

3. Conical set and renormalization

In the following, we always suppose that X is a compact Riemann surface and that $f \in \mathcal{A}(X)$ is a Ahlfors islands map. We also fix a metric of constant curvature on X—the spherical

metric if $X = \hat{\mathbb{C}}$, a flat metric (say of area 1) if X is a torus, and the hyperbolic metric otherwise. A disk of radius $r > 0$ and centered at a point $z \in X$ (with respect to this metric) will be denoted by $D(z, r)$. (If r is sufficiently small, depending on X, the set $D(z, r)$ is simply connected, and hence conformally equivalent to the standard unit disk.) Euclidean disks will be denoted by $\mathbb{D}(z, r) \subset \mathbb{C}$. If $z \in \Omega \subset X$, the connected component of Ω containing z will be denoted by $\mathrm{Comp}_z(\Omega)$.

The literature contains various different definitions of the *conical Julia set*. (The term "radial" is also used interchangeably with "conical".) In many cases, the dynamical difference between the different definitions is not significant (the reader is referred to the discussion given in [P] and also in [RvS]). We shall use both the *unbranched* conical set $\mathcal{J}_c(f)$ and the *branched* conical set $\Lambda_c(f)$. We note that $\Lambda_c(f)$ is the most general definition that appears in the literature.

Definition 3.1. The d-branched conical Julia set $\mathcal{J}_c^{(d)}(f)$ is the set of points $z \in \mathcal{J}(f)$ for which there exists $\delta > 0$ and $n_j \to \infty$ such that $D_j = \mathrm{Comp}_z\left(f^{-n_j}(D(f^{n_j}(z), \delta))\right)$ is simply connected and

$$f^{n_j} : D_j \longrightarrow D(f^{n_j}(z), \delta) \tag{3.1}$$

is defined and a proper map of degree at most d.

The *unbranched conical set* is denoted by $\mathcal{J}_c(f) := \mathcal{J}_c^{(1)}(f)$. The *branched conical set* of f is defined as

$$\Lambda_c(f) := \bigcup_{d \geq 1} \mathcal{J}_c^{(d)}.$$

Every repelling periodic point belongs to $\mathcal{J}_c(f)$. In particular, $\mathcal{J}_c$ and Λ_c are dense (and forward-invariant) subsets of $\mathcal{J}(f)$. The main feature of conical points is that one can make good renormalizations, passing from small to large scales. Moreover, these "good" renormalizations characterize conical points [H, Proposition 2.3]:

Fact 3.2. *Let $z_0 \in \mathcal{J}(f)$, and let $\phi : U \to \mathbb{C}$ be a local chart in a neighborhood of z_0, with $\phi(z_0) = 0$.*

Then $z_0 \in \Lambda_c(f)$ if and only if there are integers $n_j \to \infty$ and a sequence $r_j > 0$ with $r_j \to 0$ such that

$$\Psi_j(z) = f^{n_j}(\phi^{-1}(r_j \cdot z)) \tag{3.2}$$

converges uniformly on the unit disk $\mathbb{D} \subset \mathbb{C}$ to a non-constant holomorphic map

$$\Psi : \mathbb{D} \longrightarrow \Omega = \Psi(U) \subset X.$$

Moreover, z_0 belongs to $\mathcal{J}_c(f)$ if and only if 0 is not a critical point of Ψ.

4. Conformal measures

Conformal measures were introduced by Sullivan (in analogy with the case of Kleinian groups) as natural substitutes of Lebesgue measure in the case where the Julia set has zero area.

Definition 4.1. Let $f \in \mathcal{A}(X)$ and let $\kappa > 0$, $t \geq 0$. Let m be a probability measure supported on on $\mathcal{J}(f)$.

Then m is called $\kappa\|f'\|^t$-*conformal* if, for every Borel set $E \in \mathcal{J}(f)$ for which $f_{|E}$ is injective,

$$m(f(E)) = \int_E \kappa\|f'\|^t \, dm \quad .$$

(Here the derivative is measured with respect to the natural conformal metric on X.)

The number $\log \kappa$ is most often the topological pressure. There are two important examples in the case of a rational function f. First of all, if $t = 0$ and $\kappa = deg(f)$, then the (invariant) maximal entropy measure μ_f is $deg(f)|f'|^0 = deg(f)$-conformal. The second interesting case is when $\kappa = 1$. Sullivan showed that there is always a $|f'|^t$–conformal measure for a minimal exponent $t > 0$ [Su2, Theorem 3].

Conformal measures and the conical set. Conformal measures are particularly useful when they are supported on the unbranched conical set $\mathcal{J}_c(f)$: here we can pass from small scales to large scales using univalent iterates, and conformality means that the measure behaves in a well-controlled manner under these blow-ups. We now formalize the key property we require.

Recall that a value $w \in X$ is *Fatou-exceptional* for $f \in \mathcal{A}(X)$ if the set of iterated preimages of w is finite; we denote the set of Fatou-exceptional values by $E_F(f)$. For example, ∞ is a Fatou-exceptional value for any transcendental entire function f. Let us also denote by $E_B(f)$ the set of *branch-exceptional values* for f in the sense of [RvS]; i.e. the set of points that have only finitely many *unbranched* preimages. For example, 0 and ∞ are branch-exceptional points for $f(z) = z^2$. The Ahlfors islands property implies that $E_B(f)$ and $E_F(f)$ are finite.

Definition 4.2. Let $f \in \mathcal{A}(X)$. A measure m is called $\mathcal{J}_c(f)$*-compatible* if the following hold.

(a) The topological support of m contains the Julia set $\mathcal{J}(f)$, and furthermore $m(\{z\}) = 0$ for all z.
(b) m is locally finite, except possibly near points of the branch-exceptional set $E_B(f)$.
(c) $m(\mathcal{J}_c(f)) > 0$.
(d) Let $w \in \mathbb{C} \setminus E_B(f)$, suppose that $\delta > 0$ is such that $D := D(w, \delta)$ has finite m-measure and set $\tilde{D} := D(w, \delta/2)$. Then for every $\eta > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.

Let U be a component of $f^{-n}(D)$, for some $n \geq n_0$, such that $f^n : U \to D$ is univalent, and let $\tilde{U} \subset U$ be a component of $f^{-n}(\tilde{D})$. Then for any subset $A \subset \tilde{U}$ with

$$m(A)/m(\tilde{U}) < \varepsilon,$$

we have

$$m(f^n(A))/m(\tilde{D}) < \eta.$$

Similarly, we say that m is $\Lambda_c(f)$*-compatible* if it satisfies condition (a) and furthermore:

(b') m is locally finite except possibly near points of the Fatou-exceptional set $E_F(f)$,
(c') $m(\Lambda_c(f)) > 0$,
(d') for every $\Delta > 0$, condition (d) holds with "$w \in \mathbb{C} \setminus E_B(f)$" replaced by "$w \in \mathbb{C} \setminus E_F(f)$" and "$f^n : U \to D$ is univalent" replaced by "$f^n : U \to D$ is proper of degree at most Δ".

Lemma 4.3. *Let μ be a $\kappa\|f'\|^t$-conformal measure with $\mu(\mathcal{J}_c(f)) > 0$ and $\mu(\{x\}) = 0$ for all $x \in X$. Then μ is $\mathcal{J}_c(f)$-compatible.*

The conformal measures constructed in [MyUr] *are $\mathcal{J}_c(f)$-compatible.*

Hausdorff measure of any dimension (including Lebesgue measure) is $\Lambda_c(f)$-compatible provided that $\Lambda_c(f)$ has positive and locally finite Hausdorff measure of the corresponding dimension.

Proof. Concerning property (d), it follows for $\kappa\|f'\|^t$-conformal measures directly from the Koebe distortion theorem. For Hausdorff measures property (d') can be shown by contradiction and using a renormalization argument like the one of the proof of Lemma 5.5.

The manuscript [MyUr] concerns a quite general class of hyperbolic meromorphic functions and provides, in particular, conformal probability measures associated to Hölder continuous potentials and a particular choice of Riemannian metric. Lemma 5.20 of [MyUr] yields property (d) and implies that these measures do not have any mass on points. Furthermore, Proposition 5.21 of [MyUr] shows that the support of these mesures is the set of points that do not escape to infinity which, the functions being (topologically) hyperbolic, is a subset of the unbranched conical set $\mathcal{J}_c(f)$. This shows property (c). □

Lemma 4.4. *If m is a $\mathcal{J}_c(f)$- or $\Lambda_c(f)$-compatible measure, then m is ergodic, and $\mathcal{J}_c(f)$ resp. $\Lambda_c(f)$ has full m-measure m-almost every point as a dense orbit in $J(f)$.*

(In particular, if $\Lambda_c(f)$ has positive Lebesgue measure, then $\mathcal{J}(f) = X$ and f is ergodic.)

Proof. The proof is analogous to the proof for Lebesgue measure; see [McM, RvS]. Since the result is stated in a somewhat unfamiliar framework, let us give some of the details.

Let F be a forward-invariant subset of $\mathcal{J}_c(f)$ with $m(F) > 0$. Because m is locally a finite Borel measure, we can pick a density point z_0 of F. That is, the density of F within small simply-connected sets of bounded geometry around z_0 is close to 1.

Since z_0 belongs to $\mathcal{J}_c(f)$, there is a disk $D = D(w, \delta)$ and univalent pull-backs U_j of D, $f^{n_j}(U_n) = D$, with $n_j \to \infty$ and such that $f^{n_j}(z_0) \to w$. We may assume that z_0 is not a repelling periodic point that belongs to the branch exceptional set, in which case w also does not belong to the branch-exceptional set. So we can choose δ sufficiently small that $m(D) < \infty$.

Set $\tilde{U}_j := \operatorname{Comp}_{z_0}(f^{-n_j}(\tilde{D}))$, where $\tilde{D} = D(w, \delta/2)$. Then, by choice of z_0, we have

$$m(\tilde{U}_j \cap F)/m(\tilde{U}_j) \to 1,$$

and hence, by the definition of a $\mathcal{J}_c(f)$-compatible measure, and since F is forward-invariant,

$$m(\tilde{D} \cap F)/m(\tilde{D}) \to 1.$$

So F has full measure in $\tilde{D}$.

We claim that it follows that $m(X \setminus F) = 0$. Indeed, otherwise we can (again using the density point theorem, and the fact that m does not give positive measure to singletons) find an infinite, pairwise disjoint collection of balls in each of which F does not have full measure. Using the Ahlfors islands property (or, more precisely, Lemma 2.3), at least one of these balls must have an iterated univalent preimage in D. But then F has full measure inside this iterated preimage, again by the definition of a $\mathcal{J}_c$-compatible measure. This is a contradiction.

So in particular $\mathcal{J}_c(f)$ has full m-measure and m is ergodic. Furthermore, for any open set U intersecting the Julia set, the set of points whose orbits do not enter U is forward-invariant, and hence has measure zero by the above. It follows that m-almost every orbit is dense in $J(f)$. □

In [FU], Fisher and Urbański worked with invariant Gibbs states of Hölder continuous potentials Φ satisfying the condition $\sup \Phi > P(\Phi)$, the pressure of Φ. It is well known that the Hölder continuity of Φ combined with exponential shrinking of inverse branches yields distortion estimates. The exponential shrinking of inverse branches is the object of

Mané's theorem. We would like to mention that Mané's theorem is also available for certain meromorphic functions [RvS]. Therefore we could also extend our work to such measures.

5. Invariant and univalent differentials

Let $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. A (m,n)-differential μ on the Riemann surface X is a differential that is locally of the form $\mu(z)dz^m d\bar{z}^n$. More precisely, the differential is defined by a collection of functions $\mu_i : U_i \to \mathbb{C}$, where (U_i) is a collection of local charts with corresponding local coordinates $z_i = \varphi_i(p)$, such that

$$\mu_i = \mu_j \left(\frac{dz_j}{dz_i}\right)^m \left(\frac{d\overline{z}_j}{d\overline{z}_i}\right)^n \quad \text{holds on} \quad U_i \cap U_j \, . \tag{5.1}$$

All (m,n)-differentials we consider are assumed to be non-zero by convention. We will be interested in differentials that are continuous on an open and dense subset of the Julia set, or those that are measurable with respect to a given (conformal) measure. The pullback $f^*\mu$ of a differential under a holomorphic function is well-defined except at critical points. We shall be interested in differentials that are *invariant up to a multiplicative constant*, by which we mean that there is a constant $c \in \mathbb{C}$ such that

$$f^*\mu = c\mu \tag{5.2}$$

holds where defined (for continuous differentials) resp. almost everywhere (for measurable differentials). Note that the definition implies that the support of μ is backward invariant up to the countable set of critical points resp. up to a set of measure zero.

Example 5.1. As explained in [McM, p.47], a line field can be identified with a unit Beltrami differential, i.e. a $(-1,1)$–differential whose functions μ_i have modulus one. A line field is invariant if the corresponding Beltrami differential is invariant. Examples of functions that have an invariant measurable line field (with respect to Lebesgue measure) are given by (flexible) Lattès functions and torus endomorphisms, power maps and Chebychev polynomials.

Example 5.2. All Lattès maps (even rigid ones) have differentials that are Lebesgue a.e.-invariant up to a multiplicative constant. For example, consider the case of a map that is the quotient of a linear map under the full symmetry group of a hexagonal lattice. The differential $dz^3/d\bar{z}^3$ is invariant under this symmetry group, and hence descends to a $(3,-3)$-differential (defined and continuous except at finitely many points) that is invariant up to a multiplicative constant under the quotient map.

Example 5.3. Suppose that a $(1,1)$–differential μ is f–invariant up to multiplication, where f is analytic in the complex plane. Then the invariance equation becomes $\mu \circ f|f'|^2 = c\mu$ (valid in some local charts). Taking the logarithm of this equation gives exactly the condition that $\log |f'|$ is cohomologous to a constant.

In [McM] (and [RvS]), the notion of a locally univalent line field is used. This concept has the following straightforward adaption to differentials. We say that a (m,n)–differential μ is *univalent* near a point z if there exists a neighborhood V of z and a conformal map $\psi : U \to V$ such that the pull back of μ by ψ is a constant differential, i.e.

$$\psi^*\mu = c\, dz^m\, d\overline{z}^n \quad on \quad U\, .$$

Similarly, we shall say that a differential μ is locally univalent on a set A if every $z \in A$ has a neighborhood U such that $\mu|_{A\cap U}$ agrees with a univalent differential.

We leave the proofs of the following simple fact to the reader.

Observation 5.4. *Let $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Suppose that f is a holomorphic function such that the constant differential $dz^m d\bar{z}^n$ is invariant up to a multiplicative constant. Then f is affine: $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$.*

The first step towards the rigidity theorems is a well known fact that renormalization at a conical point at which the differential is continuous in measure leads to locally univalence of the differential.

Lemma 5.5. *Let $f \in \mathcal{A}(X)$ and let $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Suppose that μ is a (m,n)-differential supported on the Julia set such that either*

(1) *μ is invariant up to a multiplicative constant and $\mu|_{\mathcal{J}(f)}$ is continuous at some point of $\Lambda_c(f)$, or*
(2) *μ is measurable with respect to some $\mathcal{J}_c(f)$-compatible or $\Lambda_c(f)$-compatible measure ν, and μ is invariant up to a multiplicative constant ν-a.e.*

Then μ is univalent on a nonempty and relatively open subset of $\mathcal{J}(f)$.

Proof. This is a standard fact, proved by renormalizing at a conical point z_0 at which the differential is continuous (resp. continuous in measure). So, let z_0 be such a point and let

$$\Psi_j = f^{n_j} \circ \alpha_j \to \Psi : \mathbb{D} \to \Omega = \Psi(\mathbb{D}) \subset X \tag{5.3}$$

be the renormalization given by Fact 3.2. From the assumption that μ is continuous (in measure) at z_0, and the definition of $\mathcal{J}_c(f)$- resp. $\Lambda_c(f)$-compatible measures, it follows that the pull back $\psi^*\mu$ is a constant differential $\Psi^{-1}(\Lambda_c \cap \Omega)$ (ν-a.e.). (Compare [MM, p. 4359] and [My, p. 561].)

This shows that μ is univalent at z_0 when $z_0 \in \mathcal{J}_c$. Finally, if $z_0 \in \Lambda_c$ and if Ψ is not univalent, then we can conclude as follows. Since $\Omega \cap \mathcal{J}(f) \neq \emptyset$ and since $\mathcal{J}(f)$ is a perfect set there is a conformal restriction of Ψ whose image still intersects $\mathcal{J}(f)$. This shows that the differential is univalent near some point in the Julia set. □

6. The case of Lebesque measure: Proof of Theorem 1.1

In this section, we study the case where the function supports a differential that is invariant (up to a multiplicative constant) and univalent on a full complex neighborhood of a point of $\mathcal{J}(f)$. Indeed, the following result completely classifies the set of such functions.

Theorem 6.1. *Let $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and suppose that $f \in \mathcal{A}(X)$ has an (m,n)-differential (not necessarily supported on the Julia set) that is invariant up to a multiplicative constant and univalent on a full complex neighborhood of some point of $\mathcal{J}(f)$.*

Then $X = W_f$ is either the Riemann sphere or a torus, and f is conformally conjugate to one of the following:

(1) *A power map $z \mapsto z^j$, $j \in \mathbb{Z}$, $|j| \geq 2$;*
(2) *a Chebyshev polynomial or its negative;*
(3) *a (rigid or flexible) Lattès mapping;*
(4) *a linear toral endomorphism.*

By Lemma 5.5, this result implies Theorem 1.1, and indeed more generally it implies Theorem 1.2 when $\mathcal{J}(f) = X$. (The remaining case is considered in the next section.)

To begin the proof, suppose that f satisfies the hypotheses of the Theorem 6.1 and let μ be the invariant differential. By assumption, there is some open connected set U, intersecting the Julia set, and a univalent map $\Psi : V \to U$ such that $\Psi^*(\mu) = dz^m\, d\bar{z}^n$.

The set U contains some repelling periodic point p of f; we may assume (by postcomposing Ψ with a translation) that $\Psi^{-1}(p) = 0$. Let us pass to an iterate $g = f^k$ for which p is fixed. We recall that again $g \in \mathcal{A}(X)$; note that g also preserves μ. Consequently, the function

$$\sigma = \Psi^{-1} \circ g \circ \Psi, \tag{6.1}$$

defined locally near the origin, preserves the constant differential $dz^m d\overline{z}^n$ up to multiplication. Thus, by Observation 5.4, we have $\sigma(z) = \lambda z$, where $\lambda = g'(p)$ has modulus greater than 1. Hence Ψ linearizes g near p, and we can extend it using the functional relation (2.1) to a maximal domain $W_\Psi \subset \mathbb{C}$. By the functional relation, the pullback of μ under this extended map is a constant differential.

So far, this is exactly the same argument as in [MM]. For rational or even entire functions the map Ψ is defined globally on $\mathbb{C}$, and in [MM] it is then shown directly that Ψ is an exponential, trigonometric or elliptic function. For meromorphic non-entire and thus for general Ahlfors islands maps, it is no longer a priori clear that $W_\Psi = \mathbb{C}$. This is the point where we need new arguments in order to carry over the ideas from [MM, My]. We start with the following lemma.

Lemma 6.2. *Let X be a Riemann surface and $V \subset \mathbb{C}$ be connected, open and nonempty. Suppose that $\Psi : V \to X$ is a holomorphic map such that the pushforward of $dz^m d\overline{z}^n$ is (almost everywhere) well-defined. Then*

(DT) *for all $z_1, z_2 \in V$ with $\Psi(z_1) = \Psi(z_2)$, there exists a "deck transformation"*

$$\gamma(z) = \alpha z + \beta \quad , \quad \alpha \in \mathbb{C} \setminus \{0\}\ , \ \ \beta \in \mathbb{C}\ ,$$

such that $\gamma(z_1) = z_2$ and $\Psi \circ \gamma = \Psi$.

Remark 6.3. In the case of a line field, i.e. if $m = 1$ and $n = -1$, an easy calculation show that the numbers α for the maps in the deck transformation property (DT) are real. This has some importance in Proposition 6.4 and in the proof of Theorem 1.1.

Proof. Let us first suppose that z_2 is not a critical point of Ψ. Then we can locally define an inverse branch Ψ_2^{-1} mapping $\Psi(z_2)$ to z_2. Hence we can define near z_1 a map $\gamma = \Psi_2^{-1} \circ \Psi$. Clearly $\gamma(z_1) = z_2$. The assumption means that γ preserves the constant differential, and hence it is of the desired form. Because V is connected, the relation $\Psi \circ \gamma = \Psi$ holds on all of V by the identity theorem.

If z_2 is a critical point, then we can perturb z_1 and z_2 to nearby points z_1' and z_2' with $\Psi(z_1') = \Psi(z_2')$, and obtain a map γ with $\gamma(z_1') = z_2'$ as above. Provided the perturbation was small enough, it follows that also $\gamma(z_1) = z_2$. $\square$

Now we have essentially reduced the problem to the following proposition, which has a distinctly classical flavor.

Proposition 6.4. *Suppose that $V \subset \mathbb{C}$ is nonempty, open, and connected, that X is a compact Riemann surface and that $\Psi : V \to X$ is a holomorphic map satisfying the weak Casorati-Weierstraß property and the deck-transformation property (DT) as above.*

Then $V = \mathbb{C}$, X is either the Riemann sphere or a torus, and Ψ is one of the following, up to pre- and postcomposition by conformal isomorphisms:

1. *the exponential map* $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$*;*
2. *the cosine map* $\cos : \mathbb{C} \to \mathbb{C}$*;*
3. *a projection $\pi : \mathbb{C} \to X$, where X is a torus;*
4. *a Weierstraß $\wp$-function $\wp : \mathbb{C} \to \hat{\mathbb{C}}$.*

(5) *the map $\wp'$ or $(\wp')^2$, where $\wp$ is the Weierstraß function associated to a hexagonal lattice;*
(6) *the map $\wp^2$, where $\wp$ is the Weierstraß function associated to a square lattice.*

Furthermore, if $\alpha \in \mathbb{R}$ for the maps in the property (DT), then the last two cases do not occur.

Proof. Let Γ be the set of all mappings $\gamma(z) = \alpha z + \beta$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$ satisfying $\Psi \circ \gamma = \Psi$ and $\gamma(V) \cap V \neq \emptyset$. Because of (DT) and the weak Casorati-Weierstraß property, Γ is quite a rich set.

Claim 1. The domain V is Γ–stable. That is, $\gamma(V) = V$ for every $\gamma \in \Gamma$.

Proof. Let $\gamma \in \Gamma$. The set $U = \gamma(V) \cap V$ is an open and non-empty subset of V on which we have $\Psi = \Psi \circ \gamma^{-1}$. It follows that Ψ (and $\Psi \circ \gamma^{-1}$) have an analytic extension to the connected open set $V \cup \gamma(V)$. Hence $V \cup \gamma(V) \subset V$ since otherwise we have a contradiction to the fact that Ψ does not extend beyond V by the weak Casorati-Weierstraß property. $\triangle$

Claim 2. Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{C})$.

Proof. The fact that V is invariant under the elements of Γ immediately implies that Γ is closed under composition, and the inverse of any element of Γ again belongs to Γ by definition. Hence Γ is a subgroup of the group of non-constant affine maps $\mathbb{C} \to \mathbb{C}$.

This group is discrete because otherwise there would be a sequence $\gamma_j \in \Gamma$ such that $\gamma_j \to Id$ and this is in contradiction with the fact that $\Psi^{-1}(z)$ is a discrete subset of V for every $z \in \mathbb{C}$.

We showed that Γ is a discrete group, hence it must be a group of isometries. $\triangle$

Claim 3. We have $V = \mathbb{C}$.

Proof. Suppose that the domain V had some finite boundary point $a \in \partial V$. Let $z_0 \in V$ and let δ be such that $\mathbb{D}(z_0, 2\delta) \subset V$. By the weak Casorati-Weierstraß property, there is a point $z_2 \in \mathbb{D}(a, \delta)$ such that $\Psi(z_2) \in \Psi(\mathbb{D}(z_0, \delta))$; so $\Psi(z_2) = \Psi(z_1)$ for some $z_1 \in \mathbb{D}(z_0, \delta)$. By property (DT), there is $\gamma \in \Gamma$ with $\gamma(z_1) = z_2$, and as we have just shown, γ is an isometry. But then

$$a \in \mathbb{D}(z_2, \delta) = \gamma(\mathbb{D}(z_1, \delta)) \subset \gamma(\mathbb{D}(z_0, 2\delta)) \subset \gamma(V),$$

which contradicts Claim 1. $\triangle$

Since $V = \mathbb{C}$, Γ is a group of isometries and Ψ has infinite degree, we have the following possibilities, up to a change of coordinates (according to the classification of wallpaper groups without reflections):

(1) $\Gamma = \{z + 2\pi i k; k \in \mathbb{Z}\}$ and $\Psi = \exp$;
(2) $\Gamma = \{\pm z + 2\pi k; k \in \mathbb{Z}\}$ and $\Psi = \cos$;
(3) Γ is a lattice and Ψ is the projection to the corresponding torus;
(4) Γ is the product of a lattice and the order two subgroup $\{z \mapsto \pm z\}$, and Ψ is a Weierstraß $\wp$-function.
(5) Γ is either the full symmetry group or a rank-two subgroup of a hexagonal lattice, and $\Psi = \wp'$ or $\Psi = (\wp')^2$, where $\wp$ is the corresponding Weierstraß function.
(6) Γ is the symmetry group of a square lattice, and $\Psi = \wp^2$, where $\wp$ is the corresponding Weierstraß function.

In the latter two cases, the group Γ contains rotations of order greater than two, and hence these cannot occur if all deck transformations are of the form $\alpha z + \beta$ with $\alpha \in \mathbb{R} \setminus \{0\}$. $\square$

Conclusion of the proof of Theorem 6.1. Let Ψ be the Poincaré function as above, and $V = W_\Psi$. Then it follows that $V = W_\Psi = \mathbb{C}$, and that Ψ is one of the maps in the finite list above. Since Ψ semi-conjugates a linear map to g, this proves the theorem for the iterate g of f.

For the details that the same is true of f, we again refer to [MM] since, as soon as we get $V = \mathbb{C}$, we are back to the situation of a global map Ψ. In fact, one uses the global map ψ to lift a convenably chosen restriction of f. This lift again preserves a constant differential up to multiplication, from which follows that the lift is affine. It follows that Ψ semi-conjugates this affine map and f which implies that f is of the desired form. $\square$

Proof of Theorem 1.1. Suppose that f has an invariant line field a.e. on a set of positive Lebesgue measure. By Lemma 5.5, this line field is univalent in a neighborhood of some point in the Julia set of f. So it follows from Theorem 6.1 combined with Remark 6.3 that we either have the desired conclusion, or f is a power map or a Chebyshev polynomial. However, the latter maps have Julia sets of zero measure (a circle or a line segment, respectively), so we are done. $\square$

Further remarks. We note that we have not used in the proof of Theorem 1.1 the full strength of the Ahlfors islands property. What really is needed is the weak Casorati-Weierstraß property combined with the density of repellers in the argument just before (6.1). The latter is not known for Casorati-Weierstraß functions. The result is nevertheless true in this more general setting since we now explain how the density of repellers can be avoided in that proof.

Theorem 6.5. *Theorem 1.1 is true for every function $f : W_f \to X$ that has the weak Casorati-Weierstraß property. Similarly, Theorem 6.1 holds for such functions if "some point of $\mathcal{J}(f)$" is replaced by "some point of $\Lambda_c(f)$".*

Proof. We have to inspect the situation given in (5.3) more closely. So let again $\Psi = \lim_{j\to\infty} f^{n_j} \circ \alpha_j : \mathbb{D} \to \Omega$ be a limit of the renormalization at the conical point $z_0 \in \Lambda_c$, where z_0 is chosen to be a Lebesgue continuity point (for the case of Theorem 1.1) or a point near which the differential μ is univalent. In both cases, it follows as in Lemma 5.5 that the pullback of μ under Ψ is constant. It suffices to show that f has a repelling periodic point in Ω.

We can choose $V \subset \Omega$ to be simply connected such that $\Psi(0) \in V$ and such that $\Psi : U \to V$ is a proper map of finite degree say d, where U is the connected component of $\Psi^{-1}(V)$ that contains 0.

Given j, k, we denote $g = f^{n_{j+k}-n_j}$. If j, k are big enough then $g^{-1}(V)$ has a component V_0 that is relatively compact in V and that contains $f^{n_j}(z_0) = f^{n_j} \circ \alpha_j(0)$. The map $g : V_0 \to V$ is proper.

Let U_0 be a connected component of $\Psi^{-1}(V_0)$; then $\overline{U}_0 \subset U$. Because the set of critical values of Ψ is countable, we can pick $a \in U_0$ such that $g(\Psi(a))$ is not a critical value of Ψ; let b be a preimage of $g(\Psi(a))$. We can define again locally a function σ such that $\Psi \circ \sigma = g \circ \Psi$ and $\sigma(a) = b$. Considering once more the invariant differential up to multiplication it follows that $\sigma(z) = \alpha z + \beta$ is a globally defined affine map. We can therefore set $U_1 = \sigma(U_0)$.

Claim: $U_1 = U$.

Let us admit the Claim for a moment. Then we have that U_0 is relatively compact in U_1 and it follows that the affine map $\sigma : U_0 \to U_1$ has a repelling fixed point in U_0. Then g must also have a repelling fixed point in $V_0 \subset \Omega$ and we are done.

It remains to establish the Claim. Indeed, if $z \in U_0$ such that $\sigma(z) \in \partial U$ then we immediately have a contradiction with $\Psi \circ \sigma(z) = g \circ \Psi(z) \in V$. Therefore we have that $U_1 \cap \partial U = \emptyset$. On the other hand, $U_1 \cap U \neq \emptyset$ and so $U_1 \subset U$. Moreover $\Psi = g \circ \Psi \circ \sigma^{-1} : U_1 \to V$ is a proper map and so $\partial U_1 \subset \partial U$. In conclusion we have inclusion of the domains and of their boundaries. They must be equal.

□

7. The case of conformal measures: proof of Theorem 1.2

To prove Theorem 1.2, we still need to deal with the case where the Julia set might not equal the entire Riemann surface X. In this case, Lemma 5.5 still shows that the restriction of the differential μ to the Julia set is locally univalent. If the Julia set is not locally contained in an analytic curve, then it follows that the differential μ can be extended to a locally univalent invariant differential also outside the Julia set, and the result from the previous section applies. Otherwise, it is possible to see that we must have $m + n = 0$. We now make these arguments somewhat more precise.

Theorem 7.1. *Let $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and suppose that $f \in \mathcal{A}(X)$ has an (m, n)-differential μ, supported on the Julia set, that is invariant up to a multiplicative constant and univalent on a relative neighborhood of some point of $\mathcal{J}(f)$.*

Then either

(1) *f is conformally conjugate to a power map, a Chebyshev polynomial or its negative, a Lattès mapping or a toral endomorphism, or*
(2) *$m + n = 0$ and there is a finite set $A \subset \mathcal{J}(f)$ such $\mathcal{J}(f)$ is locally a 1-dimensional analytic curve near every point of $\mathcal{J}(f) \setminus A$. In particular, ∂W_f is a totally disconnected set.*

Proof. Similarly as in the previous section, let z_0 be a periodic point for f near which the restriction of the differential μ to the Julia set is univalent. We may also suppose for simplicity that z_0 is not branch-exceptional, and pass to an iterate g for which z_0 is fixed. We prove the claim for g; the conclusion for f then follows as in the previous section. Set $\lambda := g'(z_0)$ and let $\Psi : W_\Psi \to X$ be the Poincaré function at z_0. Also set $\hat{\mathcal{J}} = \Psi^{-1}(\mathcal{J}(f))$ and let ν be the univalent (m, n)-differential on $\hat{\mathcal{J}}$ obtained from μ by pullback under Ψ. Note that ν is univalent near zero by assumption, and hence univalent on all of $\hat{\mathcal{J}}$ by the functional relation (2.1) and the fact that μ is invariant up to a multiplicative constant.

The fact that ν is invariant under multiplication by λ, up to a multiplicative constant C, implies that ν is a constant differential. Indeed, we have

$$\lambda^m \bar{\lambda}^n \cdot \nu(z) = C \cdot \nu(\lambda z)$$

on $\hat{\mathcal{J}}$. First substituting $z = 0$, we see that $\nu(z) = \nu(\lambda z)$ for all $z \in \hat{\mathcal{J}}$ and, by the identity theorem, ν is locally constant near 0. Finally, the functional relation again implies that ν is constant everywhere. So without loss of generality, we can suppose that $\nu = dz^m\, d\bar{z}^n$.

Since the support of μ is contained in the Julia set we consider the following modified deck transformation condition:

(DT') for all $z_1, z_2 \in \hat{\mathcal{J}}$ with $\Psi(z_1) = \Psi(z_2)$, there exists a "deck transformation"

$$\gamma(z) = \alpha z + \beta \quad , \quad \alpha \in \mathbb{C} \setminus \{0\} \ , \ \ \beta \in \mathbb{C} \ ,$$

such that $\gamma(z_1) = z_2$ and $\Psi \circ \gamma = \Psi$.

We begin by noting that we are done whenever (DT') holds:

Claim . If (DT') holds, then the formally stronger condition (DT) is also satisfied. In particular (by Proposition 6.4) we are in Case (1) of Theorem 7.1.

Proof. We inspect the proof of Proposition 6.4, letting Γ be the group of all affine deck transformations as in (DT'). The arguments proving Claims 1 and 2 apply without any changes and yield that the domain W_Ψ is Γ–stable and that Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{C})$. Similarly, the Ahlfors islands property, and the fact that $\hat{\mathcal{J}}$ is a perfect set (compare Theorem 2.2 and Lemma 2.3) easily imply that there is a point in $\mathcal{J}(f)$ that has Ψ-preimages near every point of W_Ψ. This implies that Claim 3 also holds; i.e. $W_\Psi = \mathbb{C}$. Hence Γ is one of the finitely many groups described at the end of the Theorem.

Since Ψ semiconjugates f and $z \mapsto \lambda z$, and $\hat{\mathcal{J}}$ is completely invariant under multiplication by λ, it follows that $\lambda\Gamma \subset \Gamma$. Thus the restriction $f|_{\mathcal{J}(f)}$ has finite degree $|\lambda|$ or $|\lambda|^2$. By the Ahlfors islands property, it follows that $\partial W_f = \emptyset$—i.e., f is either a rational map or a toral endomorphism—and f itself has finite degree.

Now we can verify property (DT). Take points z_1 and z_2 with $w := \Psi(z_1) = \Psi(z_2)$; as in Lemma 6.2, we may assume that these are not critical points for Ψ. Let ψ be the branch of $\Psi^{-1} \circ \Psi$ that takes z_1 to z_2; we must show that ψ is affine. If $\Psi(z_1) \in \mathcal{J}(f)$, then we are done by (DT'). So suppose that w belongs to a component U of $\mathcal{F}(F)$. If U is not a rotation domain, then $\mathcal{P}(f) \cap U$ is either discrete in U or has at most one accumulation point, where $\mathcal{P}(f)$ is the postcritical set of f (i.e., the closure of the set of critical orbits). Hence we can choose a curve γ connecting $\Psi(z_1)$ to a point $\tilde{w} \in \mathcal{J}(f)$ such that γ does not contain any postcritical points of f. Then any branch of Ψ^{-1} can be analytically extended along γ, and hence we obtain a branch $\tilde{\psi}$ of $\Psi^{-1} \circ \Psi$ that is defined on a neighborhood of a point in the Julia set. By (DT'), the map $\tilde{\psi}$, and hence ψ, is affine.

Finally, suppose that U is a rotation domain. For $i \in \{1, 2\}$ and $k \in \mathbb{N}$, define $\zeta_i^k := \Psi(z_i/\lambda^k)$. Let k be sufficiently large that $\zeta_1^k, \zeta_2^k \notin U$, and let γ be a curve connecting w to $\mathcal{J}(f)$ such that γ is disjoint from $f^k(\mathcal{P}(f) \setminus U)$ and the set of critical values of f^k. Recalling that $\Psi(z) = f^k(\Psi(z/\lambda^k))$, we see again that the branches of Ψ^{-1} taking w to z_1 and z_2 can be continued analytically along γ, and we obtain the conclusion as above.

(Alternatively, we could have observed that $\hat{\mathcal{J}}$ is the limit of $\Gamma(0)/\lambda^n$ as $n \to \infty$, and that hence $\mathcal{J}(f)$ must contain an analytic curve. Furthermore, by the Gross star theorem [N, Page 292], every branch of Ψ^{-1} can be continued analytically along almost every radial line. Together, these facts lead to a shorter but less dynamical proof of property (DT).) △

In order to check whether (DT') holds or not, let $z_1, z_2 \in \hat{\mathcal{J}}$ and suppose that γ is a local defined holomorphic function with $\gamma(z_1) = z_2$ and such that $\Psi \circ \gamma = \Psi$. Such a map preserves the constant differential ν when restricted to $\hat{\mathcal{J}}$. If it would preserve the differential everywhere (not just on $\hat{\mathcal{J}}$), then the map is affine and we are done. Here however we have to distinguish a number of cases.

$\mathcal{J}(f)$ **is not locally contained in an analytic curve near** z_0**.** Since z_0 was assumed not to be branch-exceptional, it follows that $\mathcal{J}(f)$ cannot locally be an analytic curve anywhere, and hence the same is true for $\hat{\mathcal{J}}$. Any deck transformation γ as above satisfies

$$\gamma'(z)^m \cdot \bar{\gamma'(z)}^n = 1 \tag{7.1}$$

for all $z \in \hat{\mathcal{J}}$. By assumption, $\hat{\mathcal{J}}$ is not locally contained in an analytic curve, and hence γ' must be constant; i.e. γ is affine, as desired.

$\mathcal{J}(f)$ **is locally contained in an analytic curve near** z_0**, and** $m + n \neq 0$**.** Then $\hat{\mathcal{J}}$ is an analytic curve near 0; since $\hat{\mathcal{J}}$ is invariant under multiplication by λ, it follows that λ is real

and that $\hat{\mathcal{J}}$ is contained in a straight line through the origin; without loss of generality, we may assume that $\hat{\mathcal{J}} \subset \mathbb{R}$.

In particular, every deck transformation γ as above must preserve the real line; i.e. $\gamma'(z) \in \mathbb{R}$ for all $z \in J(f)$. Given that $m+n \neq 0$, we see that (7.1) implies that $\gamma'(z)^{m+n}$, and hence $\gamma'(z)$, is constant.

$\mathcal{J}(f)$ is locally contained in an analytic curve near z_0, and $m+n=0$. Again, we can assume that $\hat{\mathcal{J}} \subset \mathbb{R}$. Recall that Ψ is an Ahlfors islands map. Hence, if $z \in J(f)$, then $J(f)$ is locally an analytic curve near z provided that there is w with $\Psi(w) = z$ and $\Psi'(w) \neq 0$.

Because Ψ is an Ahlfors islands map, such a point w exists for all $z \in J(f)$ apart possibly from a finite set A. Furthermore, recall that the boundary ∂W_f of the domain of definition of f is contained in the Julia set of f. If ∂W_f contained a nontrivial subcontinuum, then there would be a point where ∂W_f is an analytic curve that is isolated in the Julia set, which is impossible by the Ahlfors islands property. Hence ∂W_f is totally disconnected. □

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